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Explicit Iterations with Monotonicity for Finite Element Approximations Applied to a System of Nonlinear Elliptic Equations

KAZUO ISHIHARA

Department of Mathematics, Kyushu Institute of Technology, Tobata, Kitakyushu 804, Japan

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1. INTRODUCTION

We consider explicit monotone iterations of the finite element approximations to the Dirichlet problem for the nonlinear elliptic equations:

$$\begin{aligned} \Delta u &= b_1 u^{n_1} v^{n_2}, & \Delta v &= b_2 u^{n_1} v^{n_2} & &\text{in } \Omega, \\ u &= g_1(x), & v &= g_2(x) & &\text{on } \Gamma = \partial\Omega. \end{aligned} \tag{1.1}$$

Here $x = (x_1, ..., x_n)$, Ω is a polyhedral domain in the n-dimensional Euclidean space $\mathbb{R}^n$, $\Gamma = \partial\Omega$ is the boundary of Ω, Δ is the Laplace operator, b_1, b_2 are positive constants, n_1, n_2 are positive integers, and the given functions g_1, g_2 are smooth and nonnegative. Systems of this type arise in chemical reactions [1, 4]. In such cases, u, v represent the concentrations, so that u, v are required to be nonnegative. The uniqueness and existence of the nonnegative solution for (1.1) is known [9, 10].

In a previous paper [6], we presented *implicit* iterations for solving a system of nonlinear algebraic equations. From a computational viewpoint, the disadvantage of such implicit iterations is that a set of linear equations has to be solved at each stage. In the case of a very large scale problem, it is not desirable to use implicit iterations.

The aim of this paper is to present *explicit* iterations which are the generalization of [8]. These iterations provide upper and lower bounds for the solution of the discrete problem. Moreover we give monotone convergence proof. Use of explicit iterations simplifies the program coding procedures and results in significant reduction in computational efforts. Finally some numerical results are given.

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For the finite element approximations to a single equation $\Delta u = bu^2$, we refer to [7, 8].

2. Finite Element Approximation

For given nonnegative functions g_1, g_2, we assume that

$$G_i \equiv \max\{ g_i(x); x \in \Gamma\} > 0, \qquad i = 1, 2.$$

From the maximum principle [5], the unique nonnegative solution $\{u, v\}$ of (1.1) satisfies

$$0 \leqslant u \leqslant G_1, \qquad 0 \leqslant v \leqslant G_2.$$

First, we triangulate Ω in such a way that $\bar{\Omega} = T_1 \cup T_2 \cup \cdots \cup T_J$, where T_q, $1 \leqslant q \leqslant J$, are nondegenerate closed n-simplices whose interiors are pairwise disjoint. By P_i, $1 \leqslant i \leqslant N$ (or P_i, $N+1 \leqslant i \leqslant N+M$), we denote the vertices of the triangulation which belong to Ω (or Γ). Set

$$h_q = \text{diameter of } T_q, \qquad h = \max\{h_q; 1 \leqslant q \leqslant J\},$$

$$\rho_q = \text{supremum of the diameter of the inscribed sphere of } T_q.$$

We say that a family $\{\mathscr{T}^h\}$ of triangulations is regular if there exists a positive constant c independent of the triangulation such that

$$h_q \leqslant c\rho_q \qquad \text{for all} \quad T_q \in \mathscr{T}^h.$$

For $T_q \in \mathscr{T}^h$, let $P_0^{(q)}, P_1^{(q)}, \ldots, P_n^{(q)}$ be its vertices, and let $\lambda_j^{(q)}(x)$, $0 \leqslant j \leqslant n$, be the barycentric coordinates of a point $x \in T_q$ with respect to $P_j^{(q)}$. Define

$$\sigma_q = \max\{\cos(\nabla\lambda_i^{(q)}, \nabla\lambda_j^{(q)}); 0 \leqslant i < j \leqslant n\}, \sigma = \max\{\sigma_q; 1 \leqslant q \leqslant J\}.$$

We say that a triangulation $\mathscr{T}^h$ is of acute type if $\sigma \leqslant 0$. We note that in case $n = 2$, $\mathscr{T}^h$ is of acute type if and only if all the angles of the triangles of $\mathscr{T}^h$ are less than or equal to $\pi/2$ [2].

The barycentric subdivision B_i^q of T_q corresponding to P_i which is the vertex of T_q with the barycentric coordinate $\lambda_0^{(q)}(x)$ is given by

$$B_i^q = \bigcap_{j=1}^{n} \{x \in T_q; \lambda_0^{(q)}(x) \geqslant \lambda_j^{(q)}(x)\}.$$

Then the lumped mass region $\mathscr{B}(P_i)$ is defined as follows (see Fig. 1):

$$\mathscr{B}(P_i) = \bigcup_q \{B_i^q; T_q \in \mathscr{T}^h \text{ such that } P_i \text{ is a vertex of } T_q\}.$$

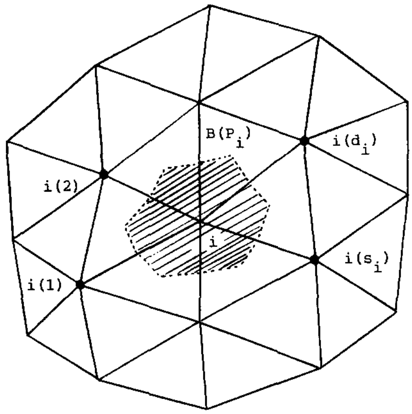

FIG. 1. Lumped mass region and nodal numbers $i(j)$.

Let $\phi_i, \bar{\phi}_i$, $1 \leqslant i \leqslant N+M$ be the finite element basis such that

$$\phi_i \text{ is continuous on } \bar{\Omega} \text{ and linear on each } T_q,$$

$$\begin{aligned} \phi_i(P_j) &= 1, \quad i = j, & \bar{\phi}_i(x) &= 1, \quad x \in \mathscr{B}(P_i), \\ &= 0, \quad i \neq j, & &= 0, \quad x \notin \mathscr{B}(P_i), \end{aligned} \qquad 1 \leqslant i, j \leqslant N+M.$$

If we seek the finite element lumped solution $\{u_h, v_h\}$ for (1.1) in the form

$$g_i^{(h)} = \sum_{j=N+1}^{N+M} g_i(P_j)\phi_j, \; i = 1, 2, \qquad u_h = \sum_{j=1}^{N} \xi_j \phi_j + g_1^{(h)},$$

$$v_h = \sum_{j=1}^{N} \zeta_j \phi_j + g_2^{(h)},$$

then $(\xi_1, ..., \xi_N, \zeta_1, ..., \zeta_N)$ satisfies the following system of nonlinear algebraic equations:

$$\begin{aligned} &H_i(\xi_1, ..., \xi_N, \zeta_1, ..., \zeta_N) \\ &\quad \equiv \sum_{j=1}^{N} a_{i,j}\xi_j + \sum_{j=N+1}^{N+M} a_{i,j}\, g_{1,j} + b_1 m_i \xi_i^{n_1} \zeta_i^{n_2} = 0, \\ &Q_i(\xi_1, ..., \xi_N, \zeta_1, ..., \zeta_N) \\ &\quad \equiv \sum_{j=1}^{N} a_{i,j}\zeta_j + \sum_{j=N+1}^{N+M} a_{i,j}\, g_{2,j} + b_2 m_i \xi_i^{n_1} \zeta_i^{n_2} = 0, \end{aligned} \tag{2.1}$$

for $1 \leqslant i \leqslant N$. Here

$$a_{i,j} = \int_\Omega \sum_{p=1}^{n} \frac{\partial \phi_i}{\partial x_p} \frac{\partial \phi_j}{\partial x_p} dx, \qquad m_i = \int_\Omega \bar{\phi}_i^2 \, dx > 0,$$

$$1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N+M, \tag{2.2}$$

$$g_{1,j} = g_1(P_j) \geqslant 0, \qquad g_{2,j} = g_2(P_j) \geqslant 0, \qquad N+1 \leqslant j \leqslant N+M. \tag{2.3}$$

In the sequel, we make the following assumption.

ASSUMPTION 1. *The triangulation $\mathscr{T}^h$ is regular and of acute type.*

3. EXPLICIT ITERATIONS

From the computational viewpoint, we present the following iteration for solving (2.1):

$$\begin{aligned} w_{i,k+1} &= w_{i,k} - \frac{H_i(w_{1,k},..., w_{N,k}, y_{1,k},..., y_{N,k})}{a_{i,i} + b_1 m_i \, y_{i,k}^{n_2} F_{n_1-1}[w_{i,k}, w_{i,k}^{\max}]}, \\ y_{i,k+1} &= y_{i,k} - \frac{Q_i(w_{1,k},..., w_{N,k}, y_{1,k},..., y_{N,k})}{a_{i,i} + b_2 m_i w_{i,k}^{n_1} F_{n_2-1}[y_{i,k}, y_{i,k}]}, \end{aligned} \tag{3.1}$$

for $1 \leqslant i \leqslant N$, $k = 0, 1, 2, ...$ Here

$$w_{i,0} = 0, \qquad y_{i,0} = G_2, \qquad 1 \leqslant i \leqslant N, \tag{3.2}$$

$$\begin{aligned} F_s[\xi, \zeta] &= \sum_{j=0}^{s} \xi^{s-j} \zeta^j, \qquad s \geqslant 1, \\ &= 1, \qquad\qquad s = 0, \end{aligned}$$

$$w_{i,k}^{\max} = \max\{w_{i(1),k}, w_{i(2),k},..., w_{i(d_i),k}, w_{i(d_i+1),k},..., w_{i(s_i),k}\}, \tag{3.3}$$

$i(1), i(2),..., i(d_i)$, $i(d_i+1),..., i(s_i)$ are nodal numbers of the vertices $P_{i(1)}, P_{i(2)},..., P_{i(d_i)}, P_{i(d_i+1)},..., P_{i(s_i)}$ associated with P_i (see Fig. 1) such that $\overline{P_i P_{i(j)}}$, $1 \leqslant j \leqslant s_i$ are sides of some n-simplices of $\mathscr{T}^h$ and

$$i(1) < i, \; i(2) < i,..., \; i(d_i) < i, \; i(d_i+1) > i,..., \; i(s_i) > i,$$

$$a_{i,i(j)} < 0, \qquad 1 \leqslant j \leqslant s_i.$$

We may also present the following:

$$\begin{aligned} \bar{w}_{i,k+1} &= \bar{w}_{i,k} - \frac{H_i(\bar{w}_{1,k+1},..., \bar{w}_{i-1,k+1}, \bar{w}_{i,k},..., \bar{w}_{N,k}, \bar{y}_{1,k},..., \bar{y}_{N,k})}{a_{i,i} + b_1 m_i \, \bar{y}_{i,k}^{n_2} F_{n_1-1}[\bar{w}_{i,k}, \bar{w}_{i,k}^{\max}]}, \\ \bar{y}_{i,k+1} &= \bar{y}_{i,k} - \frac{Q_i(\bar{w}_{1,k+1},..., \bar{w}_{N,k+1}, \bar{y}_{1,k+1},..., \bar{y}_{i-1,k+1}, \bar{y}_{i,k},..., \bar{y}_{N,k})}{a_{i,i} + b_2 m_i \bar{w}_{i,k+1}^{n_1} F_{n_2-1}[\bar{y}_{i,k}, \bar{y}_{i,k}]}, \end{aligned} \tag{3.4}$$

for $1 \leqslant i \leqslant N$, $k = 0, 1, 2, \ldots$. Here

$$\bar{w}_{i,0} = 0, \qquad \bar{y}_{i,0} = G_2, \qquad 1 \leqslant i \leqslant N,$$

$$\bar{w}_{i,k}^{\max} = \max\{\bar{w}_{i(1),k+1}, \bar{w}_{i(2),k+1}, \ldots, \bar{w}_{i(d_i),k+1}, \bar{w}_{i(d_i+1),k}, \ldots, \bar{w}_{i(s_i),k}\}.$$

Furthermore we may use the following two iterations:

$$t_{i,k+1} = t_{i,k} - \frac{H_i(t_{1,k}, \ldots, t_{N,k}, z_{i,k}, \ldots, z_{N,k})}{a_{i,i} + b_1 m_i z_{i,k}^{n_2} F_{n_1-1}[t_{i,k}, G_1]},$$
$$z_{i,k+1} = z_{i,k} - \frac{Q_i(t_{1,k}, \ldots, t_{N,k}, z_{1,k}, \ldots, z_{N,k})}{a_{i,i} + b_2 m_i t_{i,k}^{n_1} F_{n_2-1}[z_{i,k}, G_2]}, \tag{3.5}$$

$$\bar{t}_{i,k+1} = \bar{t}_{i,k} - \frac{H_i(\bar{t}_{1,k+1}, \ldots, \bar{t}_{i-1,k+1}, \bar{t}_{i,k}, \ldots, \bar{t}_{N,k}, \bar{z}_{1,k}, \ldots, \bar{z}_{N,k})}{a_{i,i} + b_1 m_i \bar{z}_{i,k}^{n_2} F_{n_1-1}[\bar{t}_{i,k}, G_1]},$$
$$\bar{z}_{i,k+1} = \bar{z}_{i,k} - \frac{Q_i(\bar{t}_{1,k+1}, \ldots, \bar{t}_{N,k+1}, \bar{z}_{1,k+1}, \ldots, \bar{z}_{i-1,k+1}, \bar{z}_{i,k}, \ldots, \bar{z}_{N,k})}{a_{i,i} + b_2 m_i \bar{t}_{i,k+1}^{n_1} F_{n_2-1}[\bar{z}_{i,k}, G_2]}, \tag{3.6}$$

for $1 \leqslant i \leqslant N$, $k = 0, 1, 2, \ldots$. Here

$$t_{i,0} = \bar{t}_{i,0} = 0, \qquad z_{i,0} = \bar{z}_{i,0} = G_2, \qquad 1 \leqslant i \leqslant N.$$

4. Monotone Convergence Results

We give the monotone convergence proof. Some lemmas are prepared.

Lemma 1. *It holds that*

$$(\xi F_s[\xi, \zeta] - \xi^{s+1}) = (\zeta F_s[\xi, \zeta] - \zeta^{s+1}),$$
$$\xi^{n_1}\zeta^{n_2} - \bar{\xi}^{n_1}\bar{\zeta}^{n_2} = \zeta^{n_2} F_{n_1-1}[\xi, \bar{\xi}](\xi - \bar{\xi}) + \bar{\xi}^{n_1} F_{n_2-1}[\zeta, \bar{\zeta}](\zeta - \bar{\zeta}).$$

Proof. In case $s = 0$, the proof is immediate. When $s \geqslant 1$, we have

$$(\xi F_s[\xi, \zeta] - \xi^{s+1}) = \xi^s\zeta + \xi^{s-1}\zeta^2 + \cdots + \xi^2\zeta^{s-1} + \xi\zeta^s = (\zeta F_s[\xi, \zeta] - \zeta^{s+1}),$$
$$\xi^{n_1}\zeta^{n_2} - \bar{\xi}^{n_1}\bar{\zeta}^{n_2} = \zeta^{n_2}(\xi^{n_1} - \bar{\xi}^{n_1}) + \bar{\xi}^{n_1}(\zeta^{n_2} - \bar{\zeta}^{n_2})$$
$$= \zeta^{n_2} F_{n_1-1}[\xi, \bar{\xi}](\xi - \bar{\xi}) + \bar{\xi}^{n_1} F_{n_2-1}[\zeta, \bar{\zeta}](\zeta - \bar{\zeta}).$$

Lemma 2 [2, 6]. *Under Assumption* 1, $a_{i,j}$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N+M$, *of* (2.2) *satisfy*

$$a_{i,i} > 0,\ a_{i,j} \leqslant 0,\ a_{i,p} = 0,$$
$$i \neq j,\ p \neq i,\ p \neq i(r),\ 1 \leqslant r \leqslant s_i,\ 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N+M,$$
$$\sum_{j=1}^{N+M} a_{i,j} = \sum_{j=1}^{s_i} a_{i,i(j)} + a_{i,i} = 0, \qquad 1 \leqslant i \leqslant N.$$

LEMMA 3 [6]. *Under Assumption* 1, *there exists a unique nonnegative solution* $(\xi_1, \ldots, \xi_N, \zeta_1, \ldots, \zeta_N)$ *of* (2.1) *satisfying* $0 \leqslant \xi_i \leqslant G_1$, $0 \leqslant \zeta_i \leqslant G_2$, $1 \leqslant i \leqslant N$.

LEMMA 4. *Under Assumption* 1, *the iteration* (3.1) *satisfies*

$$w_{i,k} \geqslant 0,\ y_{i,k} \geqslant 0,\ w_{i,k+1} \leqslant w_{i,k}^{\max}, \qquad 1 \leqslant i \leqslant N,\ k = 0, 1, 2, \ldots.$$

Proof. By (3.2) we get $w_{i,0} \geqslant 0$, $y_{i,0} \geqslant 0$, $1 \leqslant i \leqslant N$. Assume that

$$w_{i,k} \geqslant 0, \qquad y_{i,k} \geqslant 0, \qquad 1 \leqslant i \leqslant N. \tag{4.1}$$

Then, by (3.1), (3.3), (4.1), (2.2), (2.3), and Lemma 2, we obtain

$$(a_{i,i} + b_1 m_i y_{i,k}^{n_2} F_{n_1-1}[w_{i,k}, w_{i,k}^{\max}])\, w_{i,k+1}$$
$$= -\sum_{\substack{j=1 \\ j \neq i}}^{N} a_{i,j} w_{j,k} - \sum_{j=N+1}^{N+M} a_{i,j}\, g_{1,j} + b_1 m_i y_{i,k}^{n_2}(w_{i,k} F_{n_1-1}[w_{i,k}, w_{i,k}^{\max}] - w_{i,k}^{n_1}) \geqslant 0,$$

$$(a_{i,i} + b_2 m_i w_{i,k}^{n_1} F_{n_2-1}[y_{i,k}, y_{i,k}])\, y_{i,k+1} \geqslant 0, \qquad 1 \leqslant i \leqslant N.$$

Thus, by induction we obtain (4.1), $k = 0, 1, 2, \ldots$. On the other hand, from Lemmas 1 and 2, it follows that

$$w_{i,k+1} = \frac{-\sum_{j=1}^{s_i} a_{i,i(j)} w_{i(j),k} + b_1 m_i y_{i,k}^{n_2}(w_{i,k} F_{n_1-1}[w_{i,k}, w_{i,k}^{\max}] - w_{i,k}^{n_1})}{a_{i,i} + b_1 m_i y_{i,k}^{n_2} F_{n_1-1}[w_{i,k}, w_{i,k}^{\max}]} \leqslant w_{i,k}^{\max},$$

for $1 \leqslant i \leqslant N$, $k = 0, 1, 2, \ldots$. Hence the proof is complete.

We are now in a position to prove the following theorem.

THEOREM 1. *Under Assumption* 1, *the iteration* (3.1) *satisfies*

$$0 \leqslant w_{i,k} \leqslant G_1, \qquad 0 \leqslant y_{i,k} \leqslant G_2, \qquad 1 \leqslant i \leqslant N,\ k = 0, 1, 2, \ldots,$$
$$w_{i,0} \leqslant w_{i,1} \leqslant \cdots \leqslant w_{i,k} \leqslant w_{i,k+1} \leqslant \cdots,$$
$$y_{i,0} \geqslant y_{i,1} \geqslant \cdots \geqslant y_{i,k} \geqslant y_{i,k+1} \geqslant \cdots, \qquad 1 \leqslant i \leqslant N,$$
$$\lim_{k\to\infty} w_{i,k} = \xi_i, \qquad \lim_{k\to\infty} y_{i,k} = \zeta_i, \qquad 1 \leqslant i \leqslant N.$$

Proof. From Lemma 4, it follows that

$$w_{i,k} \geqslant 0, \qquad y_{i,k} \geqslant 0, \qquad 1 \leqslant i \leqslant N,\ k = 0, 1, 2, \ldots. \tag{4.2}$$

From (3.2) it is clear that $w_{i,0} \leqslant G_1$, $1 \leqslant i \leqslant N$. Assume that

$$w_{i,k} \leqslant G_1, \qquad 1 \leqslant i \leqslant N. \tag{4.3}$$

Combining (4.3) and Lemma 4, we have

$$w_{i,k+1} \leqslant \max\{w_{i(1),k}, w_{i(2),k}, \ldots, w_{i(s_i),k}\} \leqslant G_1, \qquad 1 \leqslant i \leqslant N.$$

Hence by induction (4.3) holds for $k = 0, 1, 2, \ldots$. By (4.2), (3.1) with $k = 0$, (3.2), and Lemma 2, we have

$$w_{i,1} \geqslant 0 = w_{i,0}, \qquad y_{i,1} \leqslant G_2 = y_{i,0}, \qquad 1 \leqslant i \leqslant N.$$

Assume that

$$w_{i,0} \leqslant w_{i,1} \leqslant \cdots \leqslant w_{i,k}, \qquad y_{i,0} \geqslant y_{i,1} \geqslant \cdots \geqslant y_{i,k}, \qquad 1 \leqslant i \leqslant N. \quad (4.4)$$

From (3.1) with k and $k+1$, (4.4), Lemmas 1, 2, and 4, we get

$$\begin{aligned}
&(a_{i,i} + b_1 m_i y_{i,k}^{n_2} F_{n_1-1}[w_{i,k}, w_{i,k}^{\max}])(w_{i,k+1} - w_{i,k}) \\
&\quad = -\sum_{\substack{j=1 \\ j \neq i}}^{N} a_{i,j}(w_{j,k} - w_{j,k-1}) \\
&\qquad + b_1 m_i w_{i,k}^{n_1} F_{n_2-1}[y_{i,k-1}, y_{i,k}](y_{i,k-1} - y_{i,k}) \\
&\qquad + b_1 m_i y_{i,k-1}^{n_2}(w_{i,k} - w_{i,k-1})(F_{n_1-1}[w_{i,k-1}, w_{i,k-1}^{\max}] \\
&\qquad - F_{n_1-1}[w_{i,k-1}, w_{i,k}]) \geqslant 0, \\
&(a_{i,i} + b_2 m_i w_{i,k}^{n_1} F_{n_2-1}[y_{i,k}, y_{i,k}])(y_{i,k+1} - y_{i,k}) \leqslant 0, \qquad 1 \leqslant i \leqslant N.
\end{aligned}$$

Thus (4.4) holds for $k = 0, 1, 2, \ldots$, and the limits $\bar{\xi}_i = \lim_{k \to \infty} w_{i,k}$, $\bar{\zeta}_i = \lim_{k \to \infty} y_{i,k}$, $1 \leqslant i \leqslant N$, exist and satisfy

$$H_i(\bar{\xi}_1, \ldots, \bar{\xi}_N, \bar{\zeta}_1, \ldots, \bar{\zeta}_N) = 0, \quad Q_i(\bar{\xi}_1, \ldots, \bar{\xi}_N, \bar{\zeta}_1, \ldots, \bar{\zeta}_N) = 0, \quad 1 \leqslant i \leqslant N,$$

from (3.1). Hence an application of Lemma 3 leads to $\bar{\xi}_i = \xi_i$, $\bar{\zeta}_i = \zeta_i$, $1 \leqslant i \leqslant N$. Therefore this completes the proof of Theorem 1.

By the same arguments, we obtain the following results.

THEOREM 2. *Under Assumption* 1, (3.4), (3.5), (3.6) *converge monotonically, i.e.*,

$$\bar{w}_{i,k} \nearrow \xi_i, \; \bar{y}_{i,k} \searrow \zeta_i; \; t_{i,k} \nearrow \xi_i, \; z_{i,k} \searrow \zeta_i;$$
$$\bar{t}_{i,k} \nearrow \xi_i, \; \bar{z}_{i,k} \searrow \zeta_i; \qquad 1 \leqslant i \leqslant N.$$

Remark 1. We can construct the following iteration:

$$\begin{aligned}
\tilde{y}_{i,k+1} &= \tilde{y}_{i,k} - \frac{Q_i(\tilde{w}_{1,k}, \ldots, \tilde{w}_{N,k}, \tilde{y}_{1,k+1}, \ldots, \tilde{y}_{i-1,k+1}, \tilde{y}_{i,k}, \ldots, \tilde{y}_{N,k})}{a_{i,i} + b_2 m_i \tilde{w}_{i,k}^{n_1} F_{n_2-1}[\tilde{y}_{i,k}, \tilde{y}_{i,k}^{\max}]}, \\
\tilde{w}_{i,k+1} &= \tilde{w}_{i,k} - \frac{H_i(\tilde{w}_{1,k+1}, \ldots, \tilde{w}_{i-1,k+1}, \tilde{w}_{i,k}, \ldots, \tilde{w}_{N,k}, \tilde{y}_{1,k+1}, \ldots, \tilde{y}_{N,k+1})}{a_{i,i} + b_1 m_i \tilde{y}_{i,k+1}^{n_2} F_{n_1-1}[\tilde{w}_{i,k}, \tilde{w}_{i,k}]},
\end{aligned} \quad (4.5)$$

for $1 \leqslant i \leqslant N$, $k = 0, 1, 2, \ldots$. Here $\tilde{y}_{i,0} = 0$, $\tilde{w}_{i,0} = G_1$, $1 \leqslant i \leqslant N$,

$$\tilde{y}_{i,k}^{\max} = \max\{\tilde{y}_{i(1),k+1}, \tilde{y}_{i(2),k+1}, \ldots, \tilde{y}_{i(d_i),k+1}, \tilde{y}_{i(d_i+1),k}, \ldots, \tilde{y}_{i(s_i),k}\}.$$

Then (4.5) converges monotonically, i.e.,

$$\tilde{y}_{i,k} \nearrow \zeta_i, \qquad \tilde{w}_{i,k} \searrow \xi_i, \qquad 1 \leqslant i \leqslant N.$$

Hence the above result and Theorem 2 lead to

$$\bar{w}_{i,k} \leqslant \xi_i \leqslant \tilde{w}_{i,k}, \qquad \tilde{y}_{i,k} \leqslant \zeta_i \leqslant \bar{y}_{i,k}, \qquad 1 \leqslant i \leqslant N, k = 0, 1, 2, \ldots.$$

This provides the upper and lower bounds for the solution of (2.1).

5. Numerical Examples

We show here numerical results of the explicit iterations. Let

$$\Omega_H = \{(x_1, x_2) \in \mathbb{R}^2; \sqrt{3}/4 < \sqrt{3}x_1 + x_2 < 5\sqrt{3}/4,$$
$$0 < x_2 < \sqrt{3}/2, -3\sqrt{3}/4 < x_2 - \sqrt{3}x_1 < \sqrt{3}/4\},$$
$$\Omega_S = \{(x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < 1, 0 < x_2 < 1\}.$$

Problem 1.

$$\Delta u = 2u^2 v, \qquad \Delta v = u^2 v \qquad \text{in } \Omega_S,$$
$$u = 2/(x_1 + x_2 + 0.1), \qquad v = 1/(x_1 + x_2 + 0.1) \qquad \text{on } \Gamma_S = \partial\Omega_S.$$

Problem 2.

$$\Delta u = 3uv, \qquad \Delta v = 12uv \qquad \text{in } \Omega_H,$$
$$u = 1/(x_1 + x_2 + 1)^2, \qquad v = 4/(x_1 + x_2 + 1)^2 \qquad \text{on } \Gamma_H = \partial\Omega_H.$$

Problem 3.

$$\Delta u = 3uv, \qquad \Delta v = 12uv \qquad \text{in } \Omega_S,$$
$$u = 1/(x_1 + x_2 + 1)^2, \qquad v = 4/(x_1 + x_2 + 1)^2 \qquad \text{on } \Gamma_S = \partial\Omega_S.$$

The exact solutions for Problems 1, 2, 3 are the boundary functions extended to the domain, respectively.

We divide Ω_H into uniform mesh with equilateral triangles (7, 19, 61 nodes). We also divide Ω_S into uniform mesh with right isosceles triangles (9, 25, 81 nodes). See Fig. 2. These triangulations satisfy

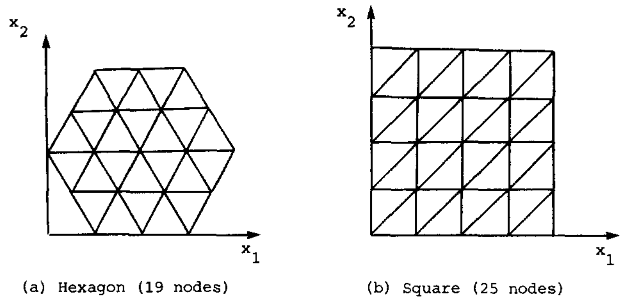

FIG. 2. Uniform mesh (acute type): (a) hexagon (19 nodes); (b) square (25 nodes).

TABLE I

Number of Iterations

Problem	Iteration	Number of iterations
Problem 1 (25 nodes)	(3.4)	27
	(3.6)	44
	(3.1)	50
	(3.5)	61
Problem 2 (19 nodes)	(3.4)	17
Problem 3 (25 nodes)	(3.4)	23

TABLE II

Numerical Results for Problem 1 with 25 Nodes at the Point $(\frac{1}{2}, \frac{1}{2})$

	Iteration (3.4)		Iteration (4.5)	
k	$\bar{w}_{i,k}$	$\bar{y}_{i,k}$	$\tilde{w}_{i,k}$	$\tilde{y}_{i,k}$
0	0.00000	10.000	20.000	0.00000
1	0.17408	8.3123	16.540	0.0028822
2	0.50598	5.2665	10.406	0.013327
3	0.84236	3.1262	6.1980	0.057920
4	1.1496	2.0374	4.0223	0.19730
5	1.3881	1.4889	2.9191	0.39946
6	1.5594	1.2131	2.3733	0.58428
10	1.8146	0.94559	1.8810	0.88626
20	1.8467	0.92348	1.8469	0.92332
26	1.8468	0.92340	1.8468	0.92340
27	1.8468	0.92340	1.8468	0.92340
	(Convergence)		(Convergence)	

Assumption 1. The numerical convergence criterion for the iteration is employed, for example, as follows:

$$\max_{1 \leqslant i \leqslant N} |w_{i,k} - w_{i,k-1}| \leqslant 10^{-6} \quad \text{and} \quad \max_{1 \leqslant i \leqslant N} |y_{i,k} - y_{i,k-1}| \leqslant 10^{-6}.$$

In Table I we show the numbers of iterations to achieve our criterion. In Table II and Fig. 3, we present the monotone convergence results. In [6] we showed that $\{u_h, v_h\}$ converges uniformly to $\{u, v\}$ as $h \to 0$. Table III

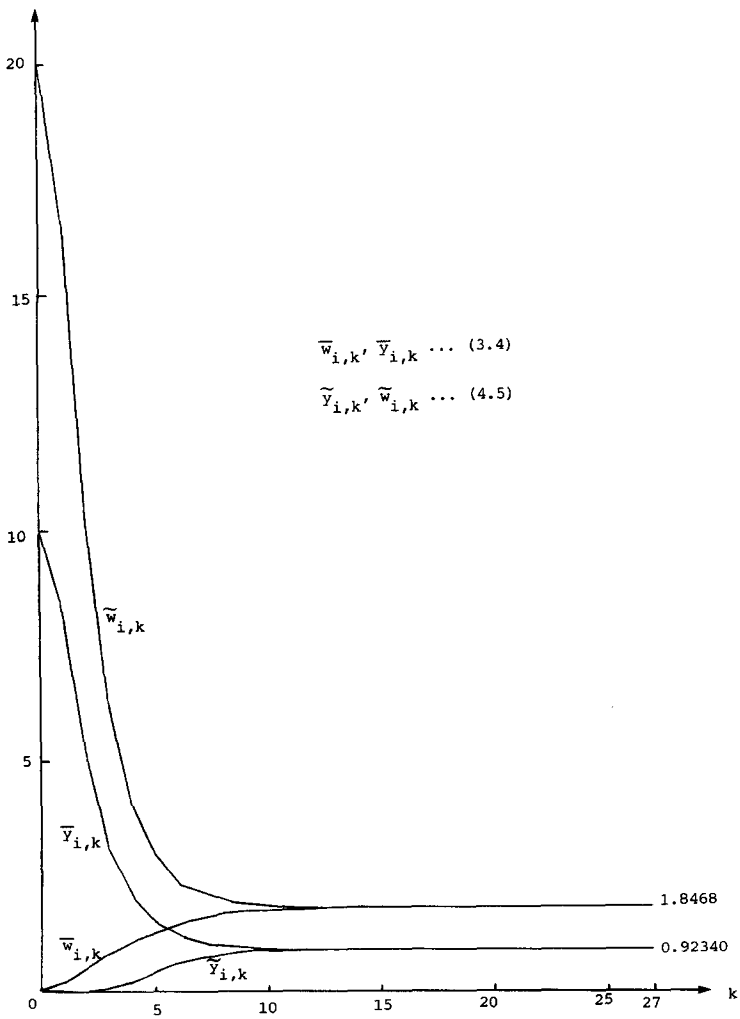

FIG. 3. Monotone convergence for Problem 1 with 25 nodes at the point $(\frac{1}{2}, \frac{1}{2})$.

TABLE III

Finite Element Solutions

(a) *Problem 1*

h	Number of nodes	$\xi_i = u_h(\frac{1}{2}, \frac{1}{2})$	$\zeta_i = v_h(\frac{1}{2}, \frac{1}{2})$
$\sqrt{2}/2$	9	1.8778	0.93891
$\sqrt{2}/4$	25	1.8468	0.92340
$\sqrt{2}/8$	81	1.8282	0.91411
Exact (continuous)		1.8182	0.90909

(b) *Problem 2*

		$\xi_i = u_h(\frac{1}{2}, \sqrt{3}/4)$	$\zeta_i = v_h(\frac{1}{2}, \sqrt{3}/4)$
$\frac{1}{2}$	7	0.27519	1.1007
$\frac{1}{4}$	19	0.26948	1.0779
$\frac{1}{8}$	61	0.26810	1.0724
Exact (continuous)		0.26763	1.0705

(c) *Problem 3*

		$\xi_i = u_h(\frac{1}{2}, \frac{1}{2})$	$\zeta_i = v_h(\frac{1}{2}, \frac{1}{2})$
$\sqrt{2}/2$	9	0.25388	1.0155
$\sqrt{2}/4$	25	0.25136	1.0054
$\sqrt{2}/8$	81	0.25038	1.0015
Exact (continuous)		0.25000	1.0000

gives the finite element solutions. These results for Problems 2, 3 coincide with those obtained in [6]. Our numerical examples verify the effectiveness of the iterations.

All computations were performed on the MELCOM-COSMO 800 III computer at Kyushu Institute of Technology.

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REFERENCES

1. C. M. ABLOW AND C. L. PERRY, Iterative solutions of the Dirichlet problem for $\Delta u = u^2$, *SIAM J. Appl. Math.* 7 (1959), 459–467.

2. P. G. CIARLET AND P. A. RAVIART, Maximum principle and uniform convergence for the finite element method, *Comput. Methods Appl. Mech. Engrg.* **2** (1973), 17–31.
3. L. COLLATZ, "Functional Analysis and Numerical Mathematics," Academic Press, New York, 1966.
4. P. V. DANKWERTS, "Gas Liquid Reactions," McGraw–Hill, New York, 1970.
5. D. GILBARG AND N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, New York, 1977.
6. K. ISHIHARA, On finite element schemes of the Dirichlet problem for a system of nonlinear elliptic equations, *Numer. Funct. Anal. Optim.* **3** (1981), 105–136.
7. K. ISHIHARA, Finite element approximations applied to the nonlinear boundary value problem $\Delta u = bu^2$, *Publ. Res. Inst. Math. Sci.* **18** (1982), 17–34.
8. K. ISHIHARA, Monotone explicit iterations of the finite element approximations for the nonlinear boundary value problem, *Numer. Math.* **43** (1984), 419–437.
9. C. S. KAHANE, On a system of nonlinear parabolic equations arising in chemical engineering, *J. Math. Anal. Appl.* **53** (1976), 343–358.
10. G. T. MCALLISTER, Difference methods for a nonlinear elliptic system of partial differential equations, *Quart. Appl. Math.* **23** (1966), 355–360.
11. J. M. ORTEGA AND W. C. RHEINBOLDT, "Iterative Solution of Nonlinear Equations in Several Variables," Academic Press, New York, 1970.
12. R. S. VARGA, "Matrix Iterative Analysis," Prentice–Hall, Englewood Cliffs, N. J., 1962.